Math 564: Real analysis and measure theory
Lecture 10

Recall that for a metric/topological space X, a Borel measure on X is any measure on the Borel σ-algebra $\mathcal{B}(X)$.

Theorem. Every finite Borel measure μ on a metric space X is strongly regular.

Proof. Let $\mathcal{S}$ be the collection of all μ-measurable sets $M \subseteq X$ which satisfy:

$$0 = \inf\{\mu(U \setminus M) : U \supseteq M \text{ open}\}$$
$$= \inf\{\mu(M \setminus C) : C \subseteq M \text{ closed}\}.$$

Claim (a). $\mathcal{S}$ contains all open sets.

Proof. Recall that open sets are F_σ in metric spaces, so for an open set $U \subseteq X$, we have $U = \bigcup_{n \in \mathbb{N}} C_n$ where the C_n are closed. Replacing each C_n with $\bigcup_{i \leq n} C_i$, we may that $U = \bigcup^{\uparrow}_{n \in \mathbb{N}} C_n$. But then, by monotone convergence, $\mu(U) = \lim_{n \to \infty} \mu(C_n)$. $\square$

Claim (b). $\mathcal{S}$ is an algebra.

Proof. Complement of open/closed is closed/open. Also finite unions of open/closed is open/closed. $\square$

Claim (c). $\mathcal{S}$ is closed under ctbl unions, and is hence a σ-algebra.

Proof. Let $M := \bigcup_{n \in \mathbb{N}} M_n$ where $M_n \in \mathcal{S}$. Since $\mathcal{S}$ is closed under finite unions, we may replace M_n with $\bigcup_{i \leq n} M_i$ and assume that $M = \bigcup^{\uparrow}_{n \in \mathbb{N}} M_n$.

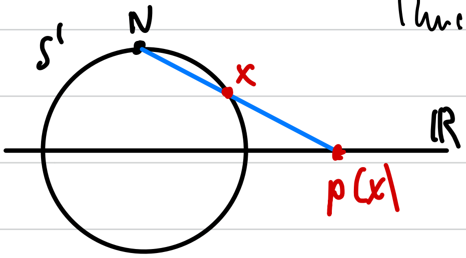
For outer regularity, let $U_n \supseteq M_n$ be open and such that $\mu(U_n \setminus M_n) \leq \varepsilon \cdot 2^{-(n+1)}$. Then $U := \bigcup_{n \in \mathbb{N}} U_n$ is open and $\mu(U \setminus M) \leq \mu\left(\bigcup_{n \in \mathbb{N}} U_n \setminus M_n\right) \leq \sum_{n \in \mathbb{N}} \mu(U_n \setminus M_n) \leq \varepsilon$.

For inner regularity, let $C_n \subseteq M_n$ be closed and such that $\mu(C_n) \approx_{1/n} \mu(M_n)$. Because $M = \bigcup^{\uparrow}_{n \in \mathbb{N}} M_n$, monotone convergence gives $\mu(M) = \lim_{n \to \infty} \mu(M_n)$, so for large enough $n \in \mathbb{N}$, we have $\mu(M) \approx_{\varepsilon/2} \mu(M_n) \approx_{\varepsilon/2} \mu(C_n)$. □

Thus, Σ contains all Borel sets. For a μ-measurable $M \subseteq X$, let $B_0 \subseteq M \subseteq B_1$ be Borel sets with $B_0 =_\mu M =_\mu B_1$ and let $U \supseteq B_1$ be open such that $\mu(U) \approx_\varepsilon \mu(B_1) = \mu(M)$ and $C \subseteq B_0$ closed such that $\mu(C) \approx_\varepsilon \mu(B_0) = \mu(M)$. So Σ contains all μ-measurable sets. □

Caution. It's not true that σ-finite Borel measures on metric spaces are regular.

Counter-example. Let $X :=$ the one-point compactification of $\mathbb{R}$, i.e. $X := \mathbb{R} \cup \{\infty\}$ with the metric of the circle S^1, which we identify with X via stereographic projection $p : S^1 \to X$ by $p(N) := \infty$ and $p : S^1 \setminus \{N\} \xrightarrow{\sim} \mathbb{R}$ as in the picture.



Thus the open sets of X are the open sets of $\mathbb{R}$ together with all sets of the form $(-\infty, a) \cup (b, \infty) \cup \{\infty\}$ (and unions of the above). Let $\bar{\lambda}$ be the Borel measure on X defined by setting $\bar{\lambda}|_{\mathbb{R}} =$ Lebesgue λ and $\bar{\lambda}(\infty) := 0$. Thus $\bar{\lambda}$ is a σ-finite Borel measure because λ is σ-finite: $X = \{\infty\} \cup \bigcup_{n \in \mathbb{N}} (-n, n)$ where each set has finite $\bar{\lambda}$-measure. And $X \cong S^1$ is obviously a metric space, in fact compact Polish. However, $\bar{\lambda}(\infty) = 0 \neq \infty = \mu(U)$ where $U \ni \infty$ is open hence $U \supseteq (-\infty, a) \cup (b, +\infty)$.

Note that in this example, X cannot be written as a ctbl union of finite measure open sets. It turns out that this is the only obstruction.

Def. Let X be a Hausdorff topological space (e.g. a metric space) and let μ be a Borel measure on X. We say that μ is

- **σ-finite by open sets** if $X = \bigcup_{n \in \mathbb{N}} U_n$ where U_n is open and has finite μ-measure.

- **finite on compact sets** if each compact set has finite μ-measure.
- **locally finite** if every point $x \in X$ admits a neighbourhood $V \ni x$ (i.e. $x \in \operatorname{int}(V)$) of finite μ-measure (in particular, an open neighbourhood of fin. meas.).

<u>Cor.</u> For a metric space X, every Borel measure that's σ-finite by open sets is strongly regular.

<u>Proof.</u> Let $X = \bigcup_{n \in \mathbb{N}} U_n$ where U_n is open and $\mu(U_n) < \infty$. Let $M \subseteq X$ be μ-measurable. For each $n \in \mathbb{N}$, viewing U_n as a metric space with $\mu|_{U_n}$ a finite Borel measure, we get an set $V_n \subseteq U_n$ open relative to U_n (hence open in X since U_n is open) such that $V_n \supseteq M \cap U_n$ and $\mu(V_n) \approx_{\varepsilon \cdot 2^{-(n+1)}} \mu(M \cap U_n)$. Thus, $V := \bigcup_{n \in \mathbb{N}} V_n$ is open in X and $\mu(V \setminus M) \leq \mu\left(\bigcup_{n \in \mathbb{N}} V_n \setminus M\right) \leq \sum_{n \in \mathbb{N}} \varepsilon \cdot 2^{-(n+1)} = \varepsilon$. This handle strong outer regularity.

For strong inner regularity, let $U \supseteq M^c$ be an open set with $\mu(U \setminus M^c) \leq \varepsilon$. But U^c is closed and $U \setminus M^c = M \setminus U^c$, hence $\mu(M \setminus U^c) \leq \varepsilon$. $\square$

Since we will use the other two conditions as well, let's sort out the relationship between them.

<u>Prop.</u> Let X be a Hausdorff topological space (e.g. a metric space). For a Borel measure μ, the following implications hold:

X is loc. compact

(1) μ is finite on compact sets.

(2) μ is locally finite.

X is 2^{nd} ctbl

(3) μ is σ-finite by open sets.

<u>Proof.</u> (3) $\Rightarrow$ (2). If $X = \bigcup_{n \in \mathbb{N}} U_n$, where U_n is open and has finite measure, then for each $x \in X$ there is $U_n \ni x$.

(2) => (1). Let K be a compact set and for each $x \in K$, let $U_x \ni x$ be an open neighbourhood of finite measure. Then the cover $\{U_x\}_{x \in K}$ of K admits a finite subcover $U_{x_1}, \dots, U_{x_n}$, so $K \subseteq \bigcup_{i \leq n} U_{x_i}$ and $\bigcup_{i \leq n} U_{x_i}$ has finite meas.

(1) => (2). Suppose X is locally compact and (1) holds. Then every point $x \in X$ has a compact neighbourhood and compact sets have finite measure.

(2) => (3). Suppose X is 2^{nd} ctbl and μ is locally finite. Let $\{U_n\}$ be a ctbl basis for X. Then for each $x \in X$ there is an open neighbourhood $U \ni x$ of finite measure, hence $\exists n_x$ with $x \in U_{n_x} \subseteq U$, so U_n has finite measure. But then $X = \bigcup_{x \in X} U_{n_x}$ and this union is ctbl. $\square$

Thus, since $\mathbb{R}^d$ and $A^{\mathbb{N}}$ (with finite A) are locally compact 2^{nd} ctbl metric spaces, all these notions coincide for Borel measures on them. In particular:

Cor. The Lebesgue measure on $\mathbb{R}^d$ and the Bernoulli measures on $A^{\mathbb{N}}$, with $|A| < \infty$, are strongly regular.

Tightness.

Def. A Borel measure μ on a Hausdorff top space (e.g. metric space) is called *tight* if for every μ-measurable set $M \subseteq X$,

$$\mu(M) = \sup\{\mu(K) : K \subseteq M \text{ compact}\}.$$

Theorem. Finite Borel measures on Polish spaces are tight.

Before proving, let's recall the equivalent definitions of compactness of metric spaces:

Theorem (Compactness in metric spaces). For a metric space X, TFAE:

(1) X is compact (every open cover has a finite subcover).
(2) X is sequentially compact (every sequence has a convergent subsequence).
(3) X is complete and totally bounded.

Cor. In a complete metric space, compact = closed and totally bounded.

Proof of the tightness theorem. Since we know that a finite Borel measure μ on a Polish space X is strongly regular, every μ-measurable set can be approximated from below by closed sets, so it is enough to show that closed sets can be approximated from below by compact sets ($\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon$). Let $C \subseteq X$ be a closed set. Since C is Polish with same metric, we may as well assume $X = C$.